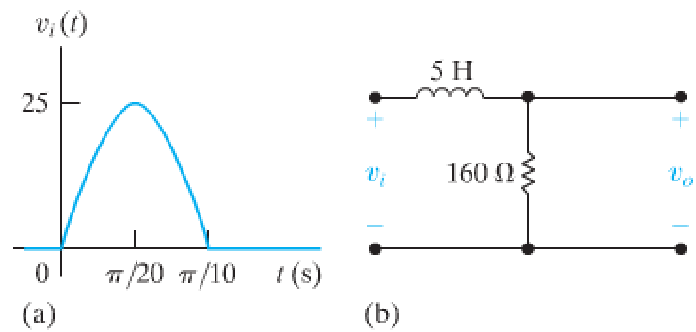
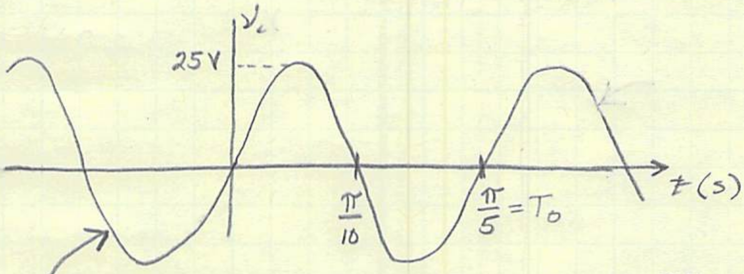
PROBLEM # 13.71

13.71 The sinusoidal voltage pulse shown in Fig. P13.71(a) is applied to the circuit shown in Fig. P13.71(b). Use the convolution integral to find the value of v_o at $t = 75$ ms.

Figure P13.71



DEVELOP A MATHEMATICAL FUNCTION FOR $v_i(t)$



$V_p = 25\,V$

$T_0 = \frac{\pi}{5}\,s$

$V_p \sin(\omega_0 t)$; $\omega_0 T_0 = 2\pi \Rightarrow \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{(\pi/5)s} = 10\,r/s$

LIMIT THE ABOVE FUNCTION TO THE LIMITS OF $V_i(t)$:

$$v_i(t) = V_p \sin(\omega_0 t)\left[u(t) - u\left(t - \frac{T_0}{2}\right)\right]$$

FIND $h(t)$

$$H(s) = \frac{Z_R}{Z_L + Z_R} \; ; \; Z_R = R; \; Z_L = sL$$

$$= \frac{R}{sL + R}$$

$$= \frac{R/L}{s + R/L}$$

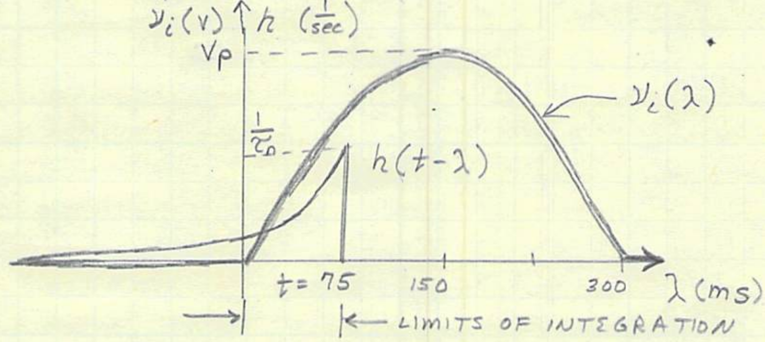
$$H(s) = \frac{1/\tau_n}{s + 1/\tau_n} \; ; \; \tau_n = L/R$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{\tau_n} e^{-\frac{t}{\tau_n}} u(t)$$

PROB 3.71 (CONT'D)

$$h(t) = \frac{1}{\tau_n} e^{-\frac{t}{\tau_n}} u(t)$$

$$h(t-\lambda) = \frac{1}{\tau_n} e^{-\frac{(t-\lambda)}{\tau_n}} u(t-\lambda)$$



EVALUATE CONVOLUTION INTEGRAL

$$v_o(t) = v_i(t) * h(t) = \int_{-\infty}^{\infty} v_i(\lambda)\, h(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} V_p \sin(\omega_0\lambda)\left[u(\lambda) - u(\lambda - \tfrac{T_0}{2})\right] \frac{1}{\tau_n} e^{-\frac{(t-\lambda)}{\tau_n}} u(t-\lambda)\, d\lambda$$

$$= \frac{V_p}{\tau_n} e^{-\frac{t}{\tau_n}} \int_{-\infty}^{\infty} \sin(\omega_0\lambda)\, e^{\frac{\lambda}{\tau_n}} \Big[\overset{\infty \;\; \boxed{t}}{u(\lambda)\, u(t-\lambda)} - \overset{\infty \;\; \boxed{t}}{u(\lambda - \tfrac{T_0}{2})\, u(t-\lambda)} \Big] d\lambda$$

BOUNDS → $\boxed{0}$ $-\infty$ $\boxed{\frac{T_0}{2}}$ $-\infty$

$$= \frac{V_p}{\tau_n} e^{-\frac{t}{\tau_n}} \left[\int_0^t \sin(\omega_0\lambda)\, e^{\frac{\lambda}{\tau_n}}\, d\tau - \int_{\frac{T_0}{2}}^{t} \sin(\omega_0\lambda)\, e^{\frac{\lambda}{\tau_n}}\, d\tau \right]$$

$$v_i(t) = \begin{cases} \dfrac{V_p}{\tau_n} e^{-\frac{t}{\tau_n}} \displaystyle\int_0^t \sin(\omega_0\lambda)\, e^{\frac{\lambda}{\tau_n}}\, d\lambda & 0 \le t \le \frac{T_0}{2} \\ \dfrac{V_p}{\tau_n} e^{-\frac{t}{\tau_n}} \displaystyle\int_0^{\frac{T}{2}} \sin(\omega_0\lambda)\, e^{\frac{\lambda}{\tau_n}}\, d\lambda & \frac{T_0}{2} \le t \end{cases}$$

ASIDE: $\int \sin(\omega_0\lambda)\, e^{\frac{\lambda}{\tau_n}}\, d\lambda = \frac{1}{2j} \int \left(e^{+j\omega_0\lambda} - e^{-j\omega_0\lambda} \right) e^{\frac{\lambda}{\tau_n}}\, d\lambda$

$$= \frac{1}{2j} \int \left(e^{(\frac{1}{\tau_n} + j\omega_0)\lambda} - e^{(\frac{1}{\tau_n} - j\omega_0)\lambda} \right) d\lambda$$

$$= \frac{1}{2j} \left[\frac{e^{(\frac{1}{\tau_n} + j\omega_0)\lambda}}{(\frac{1}{\tau_n} + j\omega_0)} - \frac{e^{(\frac{1}{\tau_n} - j\omega_0)\lambda}}{(\frac{1}{\tau_n} - j\omega_0)} \right]$$

$$= \frac{e^{\frac{\lambda}{\tau_n}}}{2j\left(\frac{1}{\tau_n^2} + \omega_0^2\right)} \left[\left(\frac{1}{\tau_n} - j\omega_0\right) e^{j\omega_0\lambda} - \left(\frac{1}{\tau_n} + j\omega_0\right) e^{-j\omega_0\lambda} \right]$$

$$= \frac{e^{\frac{\lambda}{\tau_n}}}{\frac{1}{\tau_n^2} + \omega_0^2} \left[\frac{1}{\tau_n} \frac{(e^{j\omega_0\lambda} - e^{-j\omega_0\lambda})}{2j} - j\omega_0 \cdot \frac{(e^{j\omega_0\lambda} + e^{-j\omega_0\lambda})}{2j} \right]$$

$$= \frac{e^{\frac{\lambda}{\tau_n}}}{\frac{1}{\tau_n^2} + \omega_0^2} \left[\frac{1}{\tau_n} \sin(\omega_0\lambda) - \omega_0 \cos(\omega_0\lambda) \right]$$

PROB 3.71 (CONT'D)

FOR $t = 75ms$, $T_0 = \frac{\pi}{5}s = 628ms$, $t < \frac{T_0}{2}$

$$v_i(t) = \frac{V_p}{\tau_n} e^{-\frac{t}{\tau_n}} \frac{e^{\frac{\lambda}{\tau_n}}}{\frac{1}{\tau_n^2} + \omega_0^2}\left[\frac{1}{\tau_n}\sin(\omega_0\lambda) - \omega_0\cos(\omega_0\lambda)\right]\Bigg|_{\lambda=0}^{t}$$

$$= V_p e^{-\frac{t}{\tau_n}}\left\{\frac{e^{\frac{\lambda}{\tau_n}}}{1+(\tau_n\omega_0)^2}\left[\sin(\omega_0\lambda) - (\tau_n\omega_0)\cos(\omega_0\lambda)\right]\right\}_{\lambda=0}^{t}$$

$$v_i(t) = \frac{V_p}{1+(\tau_n\omega_0)^2}\left[\sin(\omega_0 t) - (\tau_n\omega_0)\left(\cos(\omega_0 t) - e^{-\frac{t}{\tau_n}}\right)\right] \quad \left(0 \le t \le \frac{T_0}{2}\right)$$

EVALUATE RESULT AT $t = 75ms$

$$\tau_n\omega_0 = \left(\frac{L}{R}\right)\left(\frac{2\pi}{T_0}\right) = \left(\frac{5H}{160\Omega}\right)\left(10\frac{r}{s}\right) = 0.31250$$

$$(\tau_n\omega_0)^2 = (0.31250)^2 = 0.097656$$

$$\omega_0 t = (10\,r/s)(75ms) = 0.75$$

$$\frac{t}{\tau_n} = \frac{75ms}{\left(\frac{5H}{160\Omega}\right)} = 2.4$$

$$\sin(0.75) = 0.68164$$

$$\cos(0.75) = 0.73169$$

$$e^{-2.4} = 0.090718$$

$$v_o(t=75ms) = \frac{25V}{1.097656}\left[0.68164 - 0.31250(0.73169 - 0.090718)\right]$$

$$v_o(t=75ms) = 10.96\,V \leftarrow$$ (#3.71)