PROOFS OF FUNCTIONAL TRANSFORMS

FUNC TRANSFORM #1 (IMPULSE)

PROVE: $\mathcal{L}\{\delta(t)\} = 1$

$$\delta(t) = \lim_{\epsilon\to 0}\begin{cases} \frac{1}{2\epsilon} & -\epsilon < t < \epsilon \\ 0 & |t| > \epsilon \end{cases}$$

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \lim_{\epsilon\to 0}\begin{cases} \frac{1}{2\epsilon} & -\epsilon < t < \epsilon \\ 0 & |t| > \epsilon \end{cases} e^{-st}\,dt$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} e^{-st}\,dt$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\left(-\frac{1}{s}e^{-st}\right)\Big|_{-\epsilon}^{\epsilon}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\left(-\frac{1}{s}\right)\left(e^{-s\epsilon} - e^{+s\epsilon}\right); \quad e^{x} \approx 1+x,\ |x| \approx 0$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\epsilon s}\left(\epsilon s + \epsilon s\right)$$

$$= \lim_{\epsilon\to 0} \frac{\epsilon}{\epsilon}$$

$$\underline{\underline{\mathcal{L}\{\delta(t)\} = 1}} \longleftarrow \quad (\#1)$$

FUNC TRANSFORM #2 (step)

PROVE: $\mathcal{L}\{u(t)\} = \frac{1}{s}$

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t)\, e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{-st}\,dt$$

$$= -\frac{1}{s} e^{-st}\Big|_{0}^{\infty}$$

$$= -\frac{1}{s}(0-1)$$

$$\underline{\underline{\mathcal{L}\{u(t)\} = \frac{1}{s}}} \longleftarrow \quad (\#2)$$

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PROOFS OF FUNCTIONAL TRANSFORMS (CONT'D)

FUNC TRANSFORM #3 (ramp)

PROVE: $\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\mathcal{L}\{t\} = \int_{0^-}^{\infty} t\, e^{-st}\, dt \qquad \begin{array}{ll} u = t & dv = e^{-st}dt \\ du = dt & v = -\frac{1}{s}e^{-st} \end{array}$$

$$= -\frac{t}{s}e^{-st}\Big|_{0^-}^{\infty} - \int_0^{\infty}\left(-\frac{1}{s}\right)e^{-st}\,dt$$

$$= -\frac{1}{s}\left(te^{-st}\Big|_{0^-}^{\infty}\right) + \frac{1}{s}\int_0^{\infty} e^{-st}\,dt$$

$$= -\frac{1}{s}\left(\lim_{t\to 0}\frac{\frac{dt}{dt}}{\frac{de^{st}}{dt}} - 0\right) + \frac{1}{s}\mathcal{L}\{u(t)\}$$

$$= -\frac{1}{s}\left(\overset{0}{\cancel{\frac{1}{se^{\infty}}}}\right) + \frac{1}{s}\left(\frac{1}{s}\right)$$

$$\therefore\ \mathcal{L}\{t\} = \frac{1}{s^2} \qquad (\#3)$$

FUNC TRANSFORM #4 (EXPONENTIAL)

PROVE: $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

$$\mathcal{L}\{e^{-at}\} = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} e^{-(s+a)t}\,dt$$

$$\therefore\ \mathcal{L}\{e^{-at}\} = \frac{1}{s+a} \qquad (\#4)$$

FUNC TRANSFORM #5 (sine)

PROVE: $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$

$$\mathcal{L}\{\sin(\omega t)\} = \int_{0^-}^{\infty} \sin(\omega t)\, e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2j}\, e^{-st}\,dt$$

$$= \frac{1}{2j}\int_{0^-}^{\infty} e^{-(s-j\omega)t}\,dt - \frac{1}{2j}\int_{0^-}^{\infty} e^{-(s+j\omega)t}\,dt$$

$$= \frac{1}{2j}\frac{1}{(s-j\omega)} - \frac{1}{2j}\frac{1}{(s+j\omega)}$$

$$= \frac{1}{2j}\frac{1}{(s^2+\omega^2)}\left[(s+j\omega)-(s-j\omega)\right]$$

$$= \frac{1}{2j}\frac{2j\omega}{(s^2+\omega^2)}$$

$$\therefore\ \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad (\#5)$$

PROOFS OF FUNCTIONAL TRANSFORMS (CONT'D)

FUNC TRANSFORM #6 (COSINE)

PROVE: $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$

$$\mathcal{L}\{\cos(\omega t)\} = \int_{0^-}^{\infty} \cos(\omega t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \frac{e^{j\omega t}+e^{-j\omega t}}{2}\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_{0^-}^{\infty} \left(e^{-(s-j\omega)t} + e^{-(s+j\omega)t}\right) dt$$

$$= \frac{1}{2}\int_{0^-}^{\infty} e^{-(s-j\omega)t}\, dt + \frac{1}{2}\int_{0^-}^{\infty} e^{-(s+j\omega)t}\, dt$$

$$= \frac{1}{2}\,\frac{1}{s-j\omega} + \frac{1}{2}\,\frac{1}{s+j\omega}$$

$$= \frac{1}{2}\left(\frac{s+j\omega + s-j\omega}{s^2+\omega^2}\right)$$

$$\therefore\ \underline{\underline{\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}}} \longleftarrow \quad (\#6)$$

FUNC TRANSFORM #7 (DAMPED RAMP)

PROVE: $\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$

$$\mathcal{L}\{te^{-at}\} = \int_{0^-}^{\infty} te^{-at}e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} te^{-(s+a)t}\, dt \qquad \text{(FUNC TRANSFORM \#3)}$$

$$\therefore\ \underline{\underline{\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}}} \longleftarrow \quad (\#7)$$

PROOFS OF FUNCTIONAL TRANSFORMS (CONT'D)

FUNC TRANSFORM #8 (DAMPED SINE)

PROVE: $\mathcal{L}\{e^{-at}\sin(\omega t)\} = \dfrac{\omega}{(s+a)^2+\omega^2}$

$$\mathcal{L}\{e^{-at}\sin(\omega t)\} = \int_{0^-}^{\infty} e^{-at}\sin(\omega t)\, e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \frac{1}{2j}\left(e^{+j\omega t} - e^{-j\omega t}\right) e^{-(s+a)t}\,dt$$

$$= \frac{1}{2j}\left[\int_{0^-}^{\infty} e^{-[s+a-j\omega]t}\,dt - \int_{0^-}^{\infty} e^{-[s+a+j\omega]t}\,dt\right]$$

$$= \frac{1}{2j}\left[\frac{-1}{s+a-j\omega}\, e^{-(s+a-j\omega)t}\Big|_{0^-}^{\infty} - \frac{-1}{s+a+j\omega}\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2j}\left[\frac{+1}{s+a-j\omega} + \frac{-1}{s+a+j\omega}\right]$$

$$= \frac{1}{2j}\left[\frac{+(s+a+j\omega) \mp (s+a-j\omega)}{(s+a)^2+\omega^2}\right]$$

$$= \frac{1}{2j}\,\frac{2j\omega}{(s+a)^2+\omega^2}$$

$$\therefore\ \mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s+a)^2+\omega^2} \quad \longleftarrow \quad (\#8)$$

FUNC TRANSFORM #9 (DAMPED COSINE)

PROVE: $\mathcal{L}\{e^{-at}\cos(\omega t)\} = \dfrac{s+a}{(s+a)^2+\omega^2}$

$$\mathcal{L}\{e^{-at}\cos(\omega t)\} = \int_{0^-}^{\infty} e^{-at}\cos(\omega t)\, e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) e^{-(s+a)t}\,dt$$

$$= \frac{1}{2}\int_{0^-}^{\infty}\left(e^{-(s+a-j\omega)t} + e^{-(s+a+j\omega)t}\right)dt$$

$$= \frac{1}{2}\left[\frac{-1}{s+a-j\omega}\, e^{-(s+a-j\omega)t}\Big|_{0^-}^{\infty} + \frac{-1}{s+a+j\omega}\, e^{-(s+a+j\omega)t}\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+a-j\omega} + \frac{1}{s+a-j\omega}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a-j\omega) + (s+a+j\omega)}{(s+a)^2+\omega^2}\right]$$

$$= \frac{1}{2}\,\frac{2(s+a)}{(s+a)^2+\omega^2}$$

$$\therefore\ \mathcal{L}\{e^{-at}\cos(\omega t)\} = \frac{s+a}{(s+a)^2+\omega^2} \quad \longleftarrow \quad (\#9)$$

PROOFS OF OPERATIONAL TRANSFORMS

OP. TRANSFORM #1

PROVE: $\mathcal{L}\{Kf(t)\} = KF(s)$

$$\mathcal{L}\{Kf(t)\} = \int_{0^-}^{\infty} Kf(t)e^{-st}\,dt$$

$$= K\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

$$\therefore\ \mathcal{L}\{Kf(t)\} = KF(s) \qquad (\#1)$$

OP TRANSFORM #2

PROVE: $\mathcal{L}\{f_1(t)+f_2(t)-f_3(t)+\ldots\} = F_1(s)+F_2(s)-F_3(s)+\ldots$

$$\mathcal{L}\{f_1(t)+f_2(t)-f_3(t)+\ldots\} = \int_{0^-}^{\infty}\left(f_1(t)+f_2(t)-f_3(t)+\ldots\right)e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} f_1(t)e^{-st}\,dt + \int_{0^-}^{\infty} f_2(t)e^{-st}\,dt - \int_{0^-}^{\infty} f_3(t)e^{-st}\,dt + \ldots$$

$$\therefore\ \mathcal{L}\{f_1(t)+f_2(t)-f_3(t)+\ldots\} = F_1(s)+F_2(s)-F_3(s)+\ldots \qquad (\#2)$$

OP TRANSFORM #3

PROVE: $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty}\frac{df(t)}{dt}e^{-st}\,dt \; : \; u = e^{-st} \quad dv = \frac{df(t)}{dt}dt$$

$$du = -se^{-st}dt \quad v = f(t)$$

$$= f(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} -sf(t)e^{-st}\,dt$$

$$= \left(0 - f(0^-)\right) + s\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

$$\therefore\ \mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) \qquad (\#3)$$

PROOFS OF OPERATIONAL TRANSFORMS (CONT'D)

OP. TRANSFORM #4

PROVE: $\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = \mathcal{L}\left\{\frac{d}{dt}\left(\frac{df(t)}{dt}\right)\right\}$$

$$= s\mathcal{L}\left\{\frac{df(t)}{dt}\right\} - \frac{df(0^-)}{dt}$$

$$= s\left(sF(s) - f(0^-)\right) - \frac{df(0^-)}{dt} \quad \text{(OP. TRANS \#3)}$$

$$\therefore \mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt} \longleftarrow \text{(\#4)}$$

OP. TRANSFORM #5

PROVE: $\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - \sum_{k=0}^{n-1} s^{n-(k+1)}\frac{d^kf(0^-)}{dt^k}$ (NOTE: $\frac{d^0f(t)}{dt^0} = f(t)$)

PROOF BY INDUCTION: 1) ASSUME IT HOLDS FOR $n-1$ AND SHOW IT THEN HOLDS FOR n
2) SHOW IT HOLDS FOR $n=1$

STEP 1: $\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = \mathcal{L}\left\{\frac{d}{dt}\left(\frac{d^{n-1}f(t)}{dt^{n-1}}\right)\right\}$

$$= s\mathcal{L}\left\{\frac{d^{n-1}f(t)}{dt^{n-1}}\right\} - \frac{d^{n-1}f(0^-)}{dt^{n-1}} \quad \text{(OP TRANS \#3)}$$

$$= s\left[s^{n-1}F(s) - \sum_{k=0}^{n-2} s^{n-(k+1)-1}\frac{d^kf(0^-)}{dt^k}\right] - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$$

$$= s^nF(s) - \sum_{k=0}^{n-2} s^{n-(k+1)}\frac{d^kf(0^-)}{dt^k} - s^{n-((n-1)+1)}\frac{d^{n-1}f(0^-)}{dt^{n-1}}$$

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - \sum_{k=0}^{n-1} s^{n-(k+1)}\frac{d^kf(0^-)}{dt^k} \longleftarrow \text{(\#5)}$$

STEP 2: $\mathcal{L}\left\{\frac{d^1f(t)}{dt^1}\right\} = sF(s) - f(0^-)$ (OP TRANS #3)

$$\mathcal{L}\left\{\frac{d^1f(t)}{dt^1}\right\} = sF(s) - \sum_{k=0}^{n-1} s^{n-(k+1)}\frac{d^kf(0^-)}{dt^k} \; ; \; n=1$$

PROOFS OF OPERATIONAL TRANSFORMS (CONT'D)

OP TRANSFORM #6

PROVE: $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \dfrac{F(s)}{s}$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \int_{0^-}^{\infty}\int_0^t f(x)\,dx\, e^{-st}\,dt \quad ; \quad u = \int_0^t f(x)\,dx \quad dv = e^{-st}\,dt$$

$$du = f(t)\,dt \quad v = -\frac{1}{s}e^{-st}$$

$$= -\frac{1}{s}\int_0^t f(x)\,dx\, e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\left(-\frac{1}{s}\right)e^{-st}\,dt$$

$$= (0-0) + \frac{1}{s}\int_{0^-}^{\infty} f(t)\,e^{-st}\,dt$$

$$\therefore \quad \mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{F(s)}{s} \qquad (\#6)$$

OP TRANSFORM #7

PROVE: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \quad ; \quad a>0$

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} f(t-a)u(t-a)\,e^{-st}\,dt \quad ; \quad a>0$$

$$= \int_{0^-}^{\infty} f(\tau)u(\tau)\,e^{-s(\tau+a)}\,d\tau \qquad \tau = t-a,\ d\tau = dt$$

$$= \int_{0^-}^{\infty} f(\tau)u(\tau)\,e^{-s\tau}e^{-as}\,d\tau$$

$$= e^{-as}\int_{0^-}^{\infty} f(\tau)u(\tau)\,e^{-s\tau}\,d\tau$$

$$\therefore \quad \mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \qquad (\#7)$$

OP TRANSFORM #8

PROVE: $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$

$$\mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} e^{-at}f(t)\,e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} f(t)\,e^{-(s+a)t}\,dt$$

$$\therefore \quad \mathcal{L}\{e^{-at}f(t)\} = F(s+a) \qquad (\#8)$$

PROOFS OF OPERATIONAL TRANSFORMS (CONT'D)

OP TRANSFORM #9

PROVE: $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$; $a>0$

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)e^{-st}\,dt \quad ; \quad \tau = at \; ; \; d\tau = a\,dt$$

$$= \int_{0^-}^{\infty} f(\tau)\,e^{-s\left(\frac{\tau}{a}\right)}\left(\frac{1}{a}\right)d\tau$$

$$= \frac{1}{a}\int_{0^-}^{\infty} f(\tau)\,e^{-\left(\frac{s}{a}\right)\tau}\,d\tau$$

$$\therefore \; \mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad (\#9)$$

OP TRANSFORM #10

PROVE: $\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

$$\frac{dF(s)}{ds} = \frac{\partial}{\partial s}\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \frac{\partial}{\partial s}\left(f(t)e^{-st}\right)dt$$

$$= \int_{0^-}^{\infty} f(t)\frac{\partial}{\partial s}\left(e^{-st}\right)dt$$

$$= \int_{0^-}^{\infty} f(t)(-t)e^{-st}\,dt$$

$$= -\int_{0^-}^{\infty} tf(t)\,e^{-st}\,dt$$

$$= -\mathcal{L}\{tf(t)\}$$

$$\therefore \; \mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds} \qquad (\#10)$$

PROOFS OF OPERATIONAL TRANSFORMS (CONT'D)

OP TRANSFORM #11

PROVE : $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

PROOF BY INDUCTION : 1) ASSUME IT HOLDS FOR n-1 AND SHOW IT HOLDS FOR n.
2) SHOW IT HOLDS FOR n=1

STEP 1: $\mathcal{L}\{t^n f(t)\} = \mathcal{L}\{t \cdot t^{n-1} f(t)\}$; $g(t) = t^{n-1} f(t)$

$= -\frac{dG(s)}{ds}$; $G(s) = \mathcal{L}\{t^{n-1} f(t)\}$ (OP TRANSFORM #10)

$= -\frac{d}{ds} \mathcal{L}\{t^{n-1} f(t)\}$

$= -\frac{d}{ds}\left((-1)^{n-1} \frac{d^{n-1} F(s)}{ds^{n-1}}\right)$

$= (-1)(-1)^{n-1} \cdot \frac{d}{ds} \frac{d^{n-1} F(s)}{d}$

$$\underline{\underline{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}}}$$ (#11)

STEP 2: $\mathcal{L}\{t^1 f(t)\} = -\frac{dF(s)}{ds}$

$$\underline{\underline{\mathcal{L}\{t^1 f(t)\} = (-1)^1 \frac{d^1 F(s)}{ds^1}}}$$

OP TRANSFORM #12

PROVE : $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$

$F(u) = \mathcal{L}\{f(t)\}$

$= \int_{0^-}^\infty f(t) e^{-ut}\,dt$

$\int_s^\infty F(u)\,du = \int_s^\infty \int_{0^-}^\infty f(t) e^{-ut}\,dt\,du$

$= \int_{0^-}^\infty \int_s^\infty f(t) e^{-ut}\,du\,dt$

$= \int_{0^-}^\infty \left(-\frac{1}{t} f(t) e^{-ut}\right)\Big|_s^\infty dt$

$= \int_{0^-}^\infty \left(0 - -\frac{1}{t} f(t)\right) dt$

$= \int_{0^-}^\infty \frac{1}{t} f(t)\,dt$

$= \mathcal{L}\left\{\frac{1}{t} f(t)\right\}$

$$\underline{\underline{\therefore \mathcal{L}\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty F(u)\,du}}$$ (#12)