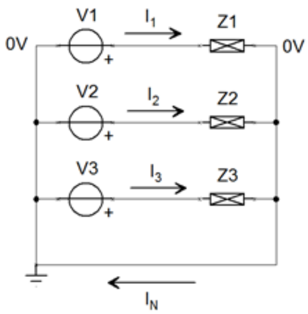
PROBLEM #1



a) If each voltage source is independent (i.e., has its own magnitude and phase) and each impedance is independent, find an expression for the phasor current $\mathbf{I_N}$ in the bottom wire?

b) If $\mathbf{V_1} = 120\text{V}\angle 35°$, $\mathbf{V_2} = 100\text{V}\angle{-55°}$, $\mathbf{V_3} = 150\text{V}\angle 165°$ and $\mathbf{Z_1} = (40+j70)\Omega$, $\mathbf{Z_2} = (20-j35)\Omega$, $\mathbf{Z_3} = 60\Omega\angle 50°$, what is $\mathbf{I_N}$?

c) If all three impedances are equal to $\mathbf{Z}$, what is the constraint that applies to the three voltages in order for $\mathbf{I_N}$ to be identically zero?

d) If, in addition to all three impedances being equal, all three voltage sources have the same magnitude, what is the constraint that applies to the three phase angles in order for $\mathbf{I_N}$ to be identically zero.

PART (a)

$$\bar{I}_N = \bar{I}_1 + \bar{I}_2 + \bar{I}_3$$

$$\bar{I}_1 = \frac{\bar{V}_1}{Z_1};\ \bar{I}_2 = \frac{\bar{V}_2}{Z_2};\ \bar{I}_3 = \frac{\bar{V}_3}{Z_3}$$

$$\therefore \bar{I}_N = \frac{\bar{V}_1}{Z_1} + \frac{\bar{V}_2}{Z_2} + \frac{\bar{V}_3}{Z_3} \longleftarrow \text{(\#1.a)}$$

PART (b)

$$\bar{I}_1 = \frac{120\text{V}\angle 35°}{(40+j70)\Omega} = \frac{120\text{V}\angle 35°}{80.62\Omega\angle 60.26°} = 1.4881\text{A}\angle{-25.26°}$$

$$\bar{I}_2 = \frac{100\text{V}\angle{-55°}}{(20-j35)\Omega} = \frac{100\text{V}\angle{-55°}}{40.31\angle{-60.26°}} = 2.481\text{A}\angle 5.256°$$

$$\bar{I}_3 = \frac{150\text{V}\angle 165°}{60\Omega\angle 50°} = 2.500\text{A}\angle 115°$$

$$\begin{aligned}
I_1 &= 1.4881\text{A}\angle{-25.26°} &&= (1.3461 - j0.6350)\text{A}\\
I_2 &= 2.481\text{A}\angle 5.256° &&= (2.4703 + j0.2272)\text{A}\\
+\ I_3 &= 2.500\text{A}\angle 115° &&= (-1.0565 + j2.2658)\text{A}\\
I_N &= &&= (2.7599 + j1.8579)\text{A}
\end{aligned}$$

$$\therefore \bar{I}_N = (2.76 + j1.858)\text{A} = 3.33\text{A}\angle 33.9° \longleftarrow \text{(\#1.b)}$$

PART (c)

$$Z_1 = Z_2 = Z_3 = Z \Rightarrow I_N = \frac{\bar{V}_1}{Z} + \frac{\bar{V}_2}{Z} = \frac{\bar{V}_3}{Z} = \frac{1}{Z}(V_1 + V_2 + V_3)$$

$$\bar{I}_N = 0 \Rightarrow \bar{V}_1 + \bar{V}_2 + \bar{V}_3 = 0 \quad (\text{or } Z = \infty) \longleftarrow \text{(\#1.c)}$$

THIS MEANS THAT THE VOLTAGE PHASORS, WHEN ADDED, MUST FORM A CLOSED POLYGON (TRIANGLE IN THIS CASE)

PROBLEM #1 (CONT'D)

PART (d)

$$|\overline{V}_1| = |\overline{V}_2| = |\overline{V}_3| = V_m$$

$$\overline{V}_1 + \overline{V}_2 + \overline{V}_3 = V_m e^{j\theta_1} + V_m e^{j\theta_2} + V_m e^{j\theta_3} = 0$$

$$e^{j\theta_1} + e^{j\theta_2} + e^{j\theta_3} = 0$$

$$e^{j\theta_1} + e^{j\theta_1} e^{j\theta_2} e^{-j\theta_1} + e^{j\theta_1} e^{j\theta_3} e^{-j\theta_1} = 0$$

$$e^{j\theta_1}\left(1 + e^{j(\theta_2-\theta_1)} + e^{j(\theta_3-\theta_1)}\right) = 0$$

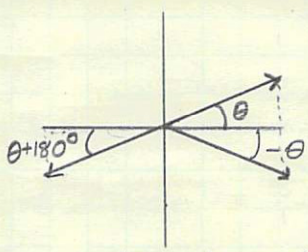
$$1 + e^{j(\theta_2-\theta_1)} + e^{j(\theta_3-\theta_1)} = 0$$

$$\underbrace{\left[1 + \cos(\theta_2-\theta_1) + \cos(\theta_3-\theta_1)\right]}_{=0} + j\underbrace{\left[\sin(\theta_2-\theta_1) + \sin(\theta_3-\theta_1)\right]}_{=0} = 0$$

$$\sin(\theta_2-\theta_1) + \sin(\theta_3-\theta_1) = 0$$

$$\sin(\theta_2-\theta_1) = -\sin(\theta_3-\theta_1)$$

$$\therefore\ (\theta_3-\theta_1) = -(\theta_2-\theta_1) \text{ or } (\theta_2-\theta_1 \pm 180^\circ)$$



$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(\theta \pm 180^\circ) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$
$$\cos(\theta \pm 180^\circ) = -\cos(\theta)$$

CASE 1: $(\theta_3-\theta_1) = -(\theta_2-\theta_1)$

$$1 + \cos(\theta_2-\theta_1) + \cos(\theta_3-\theta_1) = 1 + \cos(\theta_2-\theta_1) + \cos(\theta_2-\theta_1) = 0$$

$$1 + 2\cos(\theta_2-\theta_1) = 0 \Rightarrow \cos(\theta_2-\theta_1) = -\tfrac{1}{2} \Rightarrow \underline{(\theta_2-\theta_1) = \pm 120^\circ}$$

$$\underline{\underline{\theta_2-\theta_1 = \pm 120^\circ\ ;\ \theta_3-\theta_1 = \mp 120^\circ}} \longleftarrow \text{(\#1.c)}$$

CASE 2: $(\theta_3-\theta_1) = (\theta_2-\theta_1 \pm 180^\circ)$

$$1 + \cos(\theta_2-\theta_1) - \cos(\theta_2-\theta_1) = 0$$

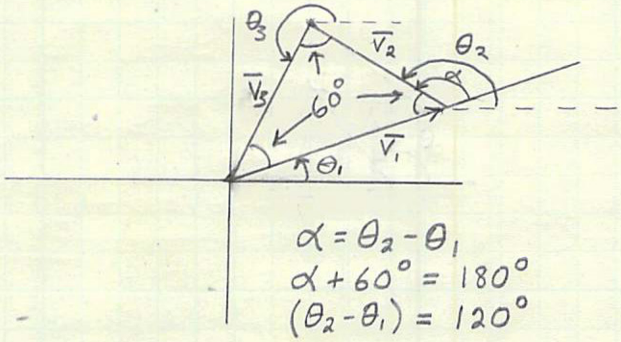
$$1 = 0 \Rightarrow \text{NO SOLUTION FOR THIS CASE}$$

PROBLEM #1 (CONT'D)

THE SOLUTION TO PART (d) ABOVE IS QUITE RIGOROUS. BUT THERE ARE OTHER WAYS TO ARRIVE AT THE RESULT.

GRAPHICALLY, WE CAN LEVERAGE THE PHASOR ADDITION RESULT MENTIONED BELOW PART (c). IF THE THREE PHASORS FORM A TRIANGLE WITH EQUAL SIDES, THEN THIS IS EQUILATERAL WITH INTERIOR ANGLES ALL 60°.

NOTE THAT INTERIOR ANGLES ARE NOT THE PHASOR ANGLES!



$$\alpha = \theta_2 - \theta_1$$
$$\alpha + 60° = 180°$$
$$(\theta_2 - \theta_1) = 120°$$

IF WE FLIP THE TRIANGLE ABOUT THE $\overline{V}_1$ AXIS, WE GET $(\theta_2 - \theta_1) = -120°$, HENCE $(\theta_2 - \theta_1) = \pm 120°$. IF WE THEN EXAMINE θ_3, WE EVENTUALLY GET $(\theta_3 - \theta_1) = \mp 120°$

ONE APPROACH THAT IS NOT CORRECT IS TO SAY

$$0 = V\angle\theta_1 + V\angle\theta_2 + V\angle\theta_3 = V(\angle\theta_1 + \angle\theta_2 + \angle\theta_3)$$

$$\therefore \theta_1 + \theta_2 + \theta_3 = 0$$

YOU CAN'T BREAK UP A PHASOR (OR ANY COMPLEX NUMBER) THIS WAY.

CONSIDER $\theta_1 = 0°,\ \theta_2 = 120°,\ \theta_3 = -120°$

YES, $\theta_1 + \theta_2 + \theta_3 = 0$ FOR THIS CASE.

BUT NOW ADD AN ANGLE β TO EACH, AND YOU GET

$$\theta_1 + \theta_2 + \theta_3 = 3\beta \neq 0 \text{ (IN GENERAL)}$$