Sufficient Conditions of Existence of Fix-Free Codes

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Abstract — **Code is fix-free if no codeword is a prefix or a suffix of any other. In this paper we improve the best-known sufficient conditions on existence of fix-free codes by a new explicit construction. We also discuss the well-known Kraft-type conjecture on the existence of fix-free codes basing on the results obtained by computer checking.**

I. Introduction

Let $C(k_1, k_2, \dots, k_n)$ denote a binary variable-length code C with k_1 codewords of length 1, k_2 codewords of length 2, $\dots$ and k_n codewords of length n.

Let $k_1, \dots, k_n$ be arbitrary positive integers. To simlify further presentation we denote $\sum\limits_{i=1}^{n} \frac{k_i}{2^i}$ by $\chi(k_1, \dots, k_n)$.

Recall that a code is called prefix-free {resp. suffix-free}, if no codeword is beginning {resp. ending} of another one. Code C, which is simultaneously prefix-free and suffix-free is called fix-free.

Our goal is to develop sufficient conditions on the existence of fix-free codes. The well-known Kraft-type conjecture [1] by R. Ahlswede is that for any values of parameters $k_1, k_2, \dots, k_n$, such that

$$\chi(k_1, k_2, \dots, k_n) \leq \frac{3}{4}, \tag{1}$$

there exists a fix-free code $C(k_1, k_2, \dots, k_n)$

The next section gives a short overview of particular cases in which the conjecture is proved.

The following lemma [1] shows that, if true, bound (1) is the best possible.

Lemma 1: For any $\varepsilon > 0$ there exist parameters $k_1, k_2, \dots, k_n$ such that $\chi(k_1, k_2, \dots, k_n) \leq \frac{3}{4} + \varepsilon$ and there exists no fix-free code $C(k_1, k_2, \dots, k_n)$.

II. Statement of Results

In this section we formulate the known sufficient conditions on existence of fix-free codes. Proofs of the next two theorems can be found in [1].

Theorem 1: If $\chi(k_1, \dots, k_n) \leq \frac{1}{2}$ then there exists a fix-free $C(k_1, \dots, k_n)$.

Theorem 2: Suppose that if $i < j$ and $k_i > 0$ and $k_j > 0$, then $i < 2j$. Then $\chi(k_1, \dots, k_n) \leq \frac{3}{4}$ implies the existence of a fix-free code $C(k_1, \dots, k_n)$.

The main result presented in current paper is formulated by the next theorem. The sketch of proof is given in section three.

Theorem 3: Let $k_1, k_2, \dots, k_n$ be arbitrary nonnegative integers. Suppose that the following statements are true:

$$\chi(k_1, \dots, k_n) \leq \frac{3}{4},$$

$$\exists p : k_1 = \dots = k_{p-1} = 0, \text{ and } \frac{k_p}{2^p} + \frac{k_{p+1}}{2^{p+1}} \geq \frac{1}{2}.$$

This implies the existence of fix-free code $C(k_1, k_2, \dots, k_n)$.

Corollary 1: If $\chi(k_1, \dots, k_n) \leq \frac{3}{4}$ and $k_1 = 1$ then there exists a fix-free code $C(k_1, k_2, \dots, k_n)$.

Note, that all the theorems formulated above are particular cases of conjecture (1). We have applied computer programming to check the conjecture for the small values of n. Thus the following theorem was obtained.

Theorem 4: Let $k_1, \dots, k_n$ be arbitrary nonnegative integers such that $\chi(k_1, \dots, k_n) \leq \frac{3}{4}$ and $n \leq 8$ then there exists a fix-free code $C(k_1, \dots, k_n)$.

One more sufficient condition of existence of fix-free codes is given in [2].

III. Sketch of the proof

We say that set $D \subseteq \{0,1\}^n$ is *left regular* {*right regular*} if all $(n-1)$-prefixes {suffixes} of words from D are pairwisely distinct.

Let $C(k_1, \dots, k_n)$ be a fix-free code. We say that a vector $w \in \{0,1\}^n$ is prefix-free {suffix-free} over C if no codeword $c \in C$ is a prefix {suffix} of w. Futher by $M(C)$ $\{\hat{M}(C)\}$ we denote the set all binary vectors of length n that are prefix-free {suffix-free} over C.

Definition: We say that fix-free code $C(k_1, \dots, k_n)$ is a π*-system* if $M(C)$ is right regular, $\hat{M}(C)$ is left regular and $\chi(k_1, \dots, k_n) = \frac{1}{2}$.

We split the proof into two sections. Firstly we study the properties and develop explicit constructions of π-systems. The main result of this section is formulated by

Lemma 2: If $k_1 = \dots = k_{n-2} = 0$, $\frac{k_{n-1}}{2^{n-1}} + \frac{k_n}{2^n} = \frac{1}{2}$, then there exists a π-system $C(k_1, k_2, \dots, k_n)$.

SecondIy we study the relationship between π-systems and fix-free codes and prove that an arbitrary π-system $C_1(k_1, \dots, k_p)$ can be extended to fix-free code $C_2(k_1, \dots, k_p + b, \dots, k_n)$, where $\chi(k_1, \dots, k_p + b, \dots, k_n) \leq \frac{3}{4}$.

References

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[2] Chunxuan Ye, Raymond W. Yeung, On Fix-Free Codes. In *Proc. ISIT 2000*, p. 426, Sorrento, Italy, June 25-30, 2000.

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